A Description of Non-Linear Wave Equations by homotopy analysis transforms method

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Abstract

In this article, a combination of Homotopy Analysis Method (HAM) and Integral Transforms (Laplace method) with less computation is proposed to solve nonlinear wave equations. Based on this method, schemes are developed to obtain approximation solutions of shock wave, soliton and travelling type solution for nonlinear wave equations. The proposed method is called Homotopy Analysis Transform Method (HATM). The results of applying this procedure to the studied cases show the high accuracy and efficiency of the new technique. The study represents the significant features of HATM also.

Keywords:

Homotopy analysis method • Integral Transform (Laplace method) • Homotopy Analysis Transform Method

AMS Classification : 65N06, 35A35

1. Introduction

The investigation of exact solutions of nonlinear wave equations plays an important role in the study of nonlinear physical phenomena. Mathematical modelling of many physical systems leads to nonlinear ordinary and partial differential equations in various fields of physics and engineering. An effective method is required to analyze the mathematical model which provides solutions conforming to physical reality. Common analytic procedures linearize the system and assume that nonlinearities are relatively insignificant. Such assumptions sometimes strongly affect the solution with respect to the real physics of the phenomenon. Thus seeking solutions of nonlinear ordinary and partial differential equations are still significant problem that needs new techniques to develop exact and approximate solutions.

Recently, many effective methods for obtaining exact solutions of nonlinear wave equations have been proposed, such as Bäcklund transformation method[1], homogeneous balance method [2,3], bifurcation method [4], Hirotas bilinear method [5], the hyperbolic tangent function expansion method [6,7], the Jacobi elliptic function expansion method [8,9], F-expansion method[10-12], Adomian decomposition method [13, 14], Homotopy analysis method [15-16], Homotopy perturbation method [17, 18], Variational iterative method [19, 20], Laplace decomposition method [21-24], modified Laplace decomposition method [25, 26]and so on

In this paper we use the homotopy analysis method combined with the Laplace transform for solving nonlinear wave equations. It is worth mentioning that the proposed method is an elegant combination of the homotopy analysis method and Laplace transform. The advantage of this proposed method is its capability of combining two powerful methods for obtaining rapid convergent series partial differential equations.

2. Basic Idea of Homotopy Analysis Method

The homotopy analysis method (HAM) is an analytical technique for solving nonlinear differential equations. HAM proposed by Liao (Liao 1992) [15], this technique is superior to the traditional perturbation methods in that it leads to convergent series solutions of strongly nonlinear problems, independent of any small or large physical parameter associated with the problem (Liao 2009) [27]. The HAM provides a more viable alternative to non perturbation techniques such as the Adomian decomposition method (ADM) (Adomian 1976; 1991) [28, 29] and other techniques that cannot guarantee the convergence of the solution series and may be only valid for weakly nonlinear problems (Liao 2009) [27]

In HAM, a system can be written as:

$$N[E(x,t)]=0 \tag{1}$$

where N is a nonlinear operator, $E(x,t)$ is unknown function of x and t, $E_0(x,t)$ is the initial guess, $\hbar \neq 0$ an auxiliary parameter and $\Re$ is a auxiliary linear operator. Also, $q \in [0,1]$ is an embedding parameter. We can construct a Homotopy as follows

$$(1-q)\Re[\phi(x,t;q)-E_0(x,t)]=q\hbar N[\phi(x,t;q)] \tag{2}$$

when q = 0, the zero-order deformation become

$$\phi(x,t;0)=E_0(x,t)$$

when $q=1$, since $\hbar \neq 0$, we get solution expression as follows

$$\phi(x,t;1)=E(x,t)$$

The embedding parameter q increases from 0 to 1. Using Taylor's theorem, $\phi(x,t;q)$ can be expanded in a power series of q as follows

$$\phi(x,t;q)=E_0(x,t)+\sum_{n=1}^{\infty}E_n(x,t)q^n \tag{3}$$

Where

$$E_n(x,t) = \frac{1}{n!} \frac{\partial^n \phi(x,t;q)}{\partial q^n}\bigg|_{q=0} \quad (4)$$

If auxiliary linear operators, the initial guesses, the auxiliary parameters, are so properly chosen, then the series (3) converges at q = 1 and

$$\phi(x,t;q) = E_0(x,t) + \sum_{n=1}^{\infty} E_n(x,t) \quad (5)$$

Differentiating (2) n times with respect to the embedding parameter q and then setting q = 0, we have the so-called n^{th} order deformation equation

$$\Re[\phi(x,t;q) - \lambda_n E_0(x,t)] = \hbar R_n\left[\vec{E}_{n-1}(x,t)\right] \quad (6)$$

Using the last equation the series solution is given by

$$E_n(x,t) = \lambda_n E_0(x,t) + \hbar L^{-1}\left\{R_n\left[\vec{E}_{n-1}(x,t)\right]\right\} \quad (7)$$

Where

$$R_n\left[\vec{E}_{n-1}(x,t)\right] = \frac{1}{(n-1)!} \frac{\partial^{n-1} N\{\phi(x,t;q)\}}{\partial q^{n-1}}\bigg|_{q=0} \quad (8)$$

And

$$\lambda_n = \begin{cases} 1 & \text{n} > 1 \\ 0 & \text{n} \geq 1 \end{cases} \quad (9)$$

2.1 Homotopy Analysis Transform Method:

We consider a general nonlinear partial differential equation

$$\mathrm{K}_i\{E(x,t)\} + \mu_j\{E(x,t)\} + N\{E(x,t)\} = 0 \quad (10)$$

Where K_i is a linear operator $\frac{\partial^i}{\partial t^i}$(i=1, 2...), μ_i is a linear operator $\frac{\partial^j}{\partial x^j}$(j=0, 1, 2...), and N is a nonlinear operator. The initial conditions are also as

$$E(x,0) = g(x) \quad E_t(x,t) = h(x)$$

Applying the Laplace transforms and we obtain *(i =2)*

$$L\{E(x,t)\} = \frac{g(x)}{p} + \frac{h(x)}{p^2} - \frac{1}{p^2}\left\{L\left[N\{E(x,t)\} + \mu_j\{E(x,t)\}\right]\right\} \quad (11)$$

Now we embed the HAM in Laplace transform method. Hence we may write non linear equation in the form

$$N\{E(x,t)\} = 0$$

$$N[\{\phi(x,t;q)\}] = L\{\phi(x,t;q)\} - \frac{g(x)}{p} - \frac{h(x)}{p^2} + \frac{1}{p^2}\{L[N\{\phi(x,t;q)\} + \mu_j\{\phi(x,t;q)\}]\} \tag{12}$$

Where N is a nonlinear operator, $E(x,t)$ is unknown function of x and t, $\hbar \neq 0$ an auxiliary parameter and $\Re$ is an auxiliary linear operator. Also, $q \in [0,1]$ is an embedding parameter. We can construct a Homotopy as follows

$$(1-q)L[\phi(x,t;q) - E_0(x,t)] = q\hbar N[\phi(x,t;q)] \tag{13}$$

when q = 0, the zero-order deformation become

$$\phi(x,t;0) = E_0(x,t)$$

when $q = 1$, since $\hbar \neq 0$, we get solution expression as follows

$$\phi(x,t;1) = E(x,t)$$

The embedding parameter q increases from 0 to 1. Using Taylor's theorem, $\phi(x,t;q)$ can be expanded in a power series of q as follows

$$\phi(x,t;q) = E_0(x,t) + \sum_{n=1}^{\infty} E_n(x,t)q^n \tag{14}$$

Where

$$E_n(x,t) = \frac{1}{n!}\frac{\partial^n \phi(x,t;q)}{\partial q^n}\bigg|_{q=0} \tag{15}$$

If auxiliary linear operators, the initial guesses, the auxiliary parameters, are so properly chosen, then the series (14) converges at q = 1 and

$$\phi(x,t;1) = E_0(x,t) + \sum_{n=1}^{\infty} E_n(x,t) \tag{16}$$

Differentiating (13) n times with respect to the embedding parameter q and then setting $q = 1$ we have the so-called nth order deformation equation

$$L[\phi(x,t;q) - \lambda_n E_0(x,t)] = \hbar R_n[\vec{E}_{n-1}(x,t)] \tag{17}$$

Using the last equation the series solution is given by

$$E_n(x,t) = \lambda_n E_{n-1}(x,t) + \hbar L^{-1}\{R_n[\vec{E}_{n-1}(x,t)]\} \tag{18}$$

Where

$$R_n[\vec{E}_{n-1}(x,t)] = \frac{1}{(n-1)!}\frac{\partial^{n-1} N\{\phi(x,t;q)\}}{\partial q^{n-1}}\bigg|_{q=0} \tag{19}$$

and

$$\lambda_n = \begin{cases} 1 & n > 1 \\ 0 & n \le 1 \end{cases} \tag{20}$$

2.3 Mathematical formation:-

Let us consider the non linear term $E\partial E/\partial x$, in the fluid equation of motion

$$\frac{\partial E}{\partial t} + E\frac{\partial E}{\partial x} = F \tag{21}$$

From the Fourier analysis techniques, a term generates mode- mode coupling and higher temporal and spatial harmonics, in real space; this means a deformation of the wave form. We lead to an eventual overtaking of the fluid elements and the breaking up of the wave.

The presence of the self-consistent field F on the right hand side often produced an effect to prohibit such an overtaking, at least within some limited time scale. If we combine the force term with the field equation, F may be expressed as a function of E also. The lowest significant linear contribution of such a term will be $\partial^2 E/\partial x^2 or\ \partial^2 E/\partial t^2$ .For a passive medium, these terms represent dissipation. If we take $\partial^2 E/\partial x^2$ as an example the equation may be written

$$\frac{\partial E}{\partial t} + E\frac{\partial E}{\partial x} - \alpha\frac{\partial^2 E}{\partial x^2} = 0 \tag{22}$$

Where α is a positive constant having a dimension of L^2/T. This equation generally called Burgers equation. As the steepening progress, the higher derivative term introduced above contributes more and when this term becomes comparable to the non linear term, the steepening is stopped.

In the absence of dissipation, the lowest significant linear contribution of the force term will be $\partial^3 E/\partial x^3$ .This term represent the lowest order *dispersion* effect. If we introduce this term into the equation (21) then

$$\frac{\partial E}{\partial t} + E\frac{\partial E}{\partial x} + \beta\frac{\partial^3 E}{\partial x^3} = 0 \tag{23}$$

Where β is a constant having a dimension of L^3/T .this called the Korteweg-de Vries (KdV) equation.

3. Application

In order to elucidate the solution procedure of the homotopy Analysis transform method (HATM), we solve two examples in this sections which shows the effectiveness and generalizations of our proposed method.

Example 1:- Consider the equation (22)

$$\frac{\partial E}{\partial t}+E\frac{\partial E}{\partial x}-\alpha\frac{\partial^2 E}{\partial x^2}=0$$

with initial condition $E(x,0)=\eta\left[1-\tanh\left(\frac{\eta x}{2\alpha}\right)\right]$

Applying Laplace transformation we have

$$L[E(x,t)]+\frac{1}{\mathrm{P}}\left\{L\left[E\frac{\partial E}{\partial x}-\alpha\frac{\partial^2 E}{\partial x^2}\right]\right\}-\frac{\eta}{\mathrm{P}}\left[1-\tanh\left(\frac{\eta x}{2\alpha}\right)\right]=0 \tag{24}$$

We define a nonlinear operator according equation (12)

$$N[\{\phi(x,t;q)\}]=L\{\phi(x,t;q)\}-\frac{\eta}{p}\left[1-\tanh\left(\frac{\eta x}{2\alpha}\right)\right]+\frac{1}{p}\left\{L\left[\phi(x,t;q)\frac{\partial\phi(x,t;q)}{\partial x}-\alpha\frac{\partial^2\phi(x,t;q)}{\partial x^2}\right]\right\} \tag{25}$$

Using above definitions, we can construct a Homotopy as follows

$$q\hbar N[\phi(x,t;q)]=(1-q)L[\phi(x,t;q)-E_0(x,t)] \tag{26}$$

Where $q\in[0,1]$, $\mathrm{E}_0(x,t)$ is an initial guess of $\mathrm{E}(x,t)$ and $\Phi(x,t;q)$ is unknown function. When $q=0$ and $q=1$ we have

$$\Phi(x,t;0)=E_0(x,t)\ ,\ \Phi(x,t;1)=E(x,t)$$

The n$^{\text{th}}$ order deformation equation is

$$E_n(x,t)=\lambda_n E_{n-1}(x,t)+\hbar L^{-1}\left\{R_n\left[\vec{E}_{n-1}(x,t)\right]\right\} \tag{27}$$

Where

$$R_n\left[\vec{E}_{n-1}(x,t)\right]=L(\mathrm{E}_{n-1})+\frac{1}{p}\left\{L\left[\sum_{k=0}^{n-1}\left[\mathrm{E}_k\frac{\partial(\mathrm{E}_{n-1-k})}{\partial x}\right]-\alpha\frac{\partial^2(\mathrm{E}_{n-1}))}{\partial x^2}\right]\right\}-(1-\lambda_n)\frac{\eta}{p}\left[1-\tanh\left(\frac{\eta x}{2\alpha}\right)\right] \tag{28}$$

Obtain the series solution (using Mathematica 5.2 package)

$$\mathrm{E}_1(x,t)=-\frac{\hbar t\eta^3}{2\alpha}\left[1-\tanh^2\left(\frac{\eta x}{2\alpha}\right)\right] \tag{29}$$

$$\mathrm{E}_2(x,t)=-\frac{\hbar t\eta^3}{2\alpha}\left[1-\tanh^2\left(\frac{\eta x}{2\alpha}\right)\right]\left\{1-\hbar+\frac{\hbar\eta^2 t}{2\alpha}\tanh\left(\frac{\eta x}{2\alpha}\right)\right\} \tag{30}$$

……..

The solution is

$$E(x,t)=\eta\left[1-\tanh\frac{\eta}{2\alpha}(x-\eta t)\right] \tag{31}$$

Where $\hbar=-1$

This expression represents shock solution with the shock speed, shock height and shock thickness given by η, η and $\alpha\eta^{-1}$ respectively. The shock solution appears because of the introduction of the dissipative term, which increases the entropy.

Example 2:- Consider the equation (23)

$$\frac{\partial E}{\partial t}+E\frac{\partial E}{\partial x}+\beta\frac{\partial^3 E}{\partial x^3}=0 \tag{32}$$

With initial condition $E(x,0)=3\eta\left[Sech^2\left(\sqrt{\frac{\eta}{\beta}}\left(\frac{x}{2}\right)\right)\right]$

Applying Laplace transformation we have

$$L(E(x,t))+\frac{1}{p}\left[L\left(E\frac{\partial E}{\partial x}+\beta\frac{\partial^3 E}{\partial x^3}\right)\right]-\frac{3\eta}{p}Sech^2\left(\sqrt{\frac{\eta}{\beta}}\left(\frac{x}{2}\right)\right)=0 \tag{33}$$

We define a nonlinear operator according equation (12)

$$N[\{\phi(x,t;q)\}]=L\{\phi(x,t;q)\}-\frac{3\eta}{p}Sech^2\left(\sqrt{\frac{\eta}{\beta}}\left(\frac{x}{2}\right)\right)+\frac{1}{p}\left\{L\left[\phi(x,t;q)\frac{\partial\phi(x,t;q)}{\partial x}+\beta\frac{\partial^3\phi(x,t;q)}{\partial x^3}\right]\right\} \tag{34}$$

The n^{th} order deformation equation is

$$E_n(x,t)=\lambda_n E_{n-1}(x,t)+\hbar L^{-1}\left\{R_n\left[\vec{E}_{n-1}(x,t)\right]\right\}$$

Where

$$R_n\left[\vec{E}_{n-1}(x,t)\right]=L(\mathrm{E}_{n-1})+\frac{1}{p}\left\{L\left[\sum_{k=0}^{n-1}\left[\mathrm{E}_k\frac{\partial(\mathrm{E}_{n-1-k})}{\partial x}\right]+\beta\frac{\partial^3(\mathrm{E}_{n-1}))}{\partial x^3}\right]\right\}-(1-\lambda_n)\frac{3\eta}{p}Sech^2\left(\sqrt{\frac{\eta}{\beta}}\left(\frac{x}{2}\right)\right) \tag{35}$$

Obtain the series solution (using Mathematica 5.2 package)

$$E_1=-3t\hbar\eta^2\sqrt{\frac{\eta}{\beta}}\left[Sech^2\left(\sqrt{\frac{\eta}{\beta}}\left(\frac{x}{2}\right)\right)\tanh\left(\sqrt{\frac{\eta}{\beta}}\left(\frac{x}{2}\right)\right)\right] \tag{36}$$

CONCLUSIONS: In this paper, the homotopy analysis transform method (HATM) is successfully applied to solve many nonlinear problems. It is apparently seen that HATM is very powerful and efficient technique in finding analytical solutions for wider class of problems. They also do not require large computer memory and discretization of variable x.

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